

# Herman's Theory Revisited (Extension)

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## Abstract

We prove that a  $C^{3+\beta}$ -smooth orientation-preserving circle diffeomorphism with rotation number in Diophantine class  $D_\delta$ ,  $0 < \beta < \delta < 1$ , is  $C^{2+\beta-\delta}$ -smoothly conjugate to a rigid rotation.

## 1 Introduction

An irrational number  $\rho$  is said to belong to Diophantine class  $D_\delta$  if there exists a constant  $C > 0$  such that  $|\rho - p/q| \geq Cq^{-2-\delta}$  for any rational number  $p/q$ .

In [1], the following result was proven.

**Theorem** (Khanin-T.). *Let  $T$  be a  $C^{2+\alpha}$ -smooth orientation-preserving circle diffeomorphism with rotation number  $\rho \in D_\delta$ ,  $0 < \delta < \alpha \leq 1$ . Then  $T$  is  $C^{1+\alpha-\delta}$ -smoothly conjugate to the rigid rotation by angle  $\rho$ .*

By the smoothness of conjugacy we mean the smoothness of the homeomorphism  $\phi$  such that

$$\phi \circ T \circ \phi^{-1} = R_\rho, \quad (1)$$

where  $R_\rho(\xi) = \xi + \rho \pmod{1}$  is the mentioned rigid rotation.

The aim of the present paper is to extend the Theorem above to the case of  $T \in C^{3+\beta}$ ,  $0 < \beta < \delta < 1$ , so that the extended result is read as follows:

**Theorem 1.** *Let  $T$  be a  $C^r$ -smooth orientation-preserving circle diffeomorphism with rotation number  $\rho \in D_\delta$ ,  $0 < \delta < 1$ ,  $2 + \delta < r < 3 + \delta$ . Then  $T$  is  $C^{r-1-\delta}$ -smoothly conjugate to the rigid rotation by angle  $\rho$ .*

Historically, the first global results on smoothness of conjugation with rotations were obtained by M. Herman [2]. Later J.-C. Yoccoz extended the theory to the case of Diophantine rotation numbers [3]. The result, recognized generally as the final answer in the theory, was proven by Y. Katznelson, D. Ornstein [4]. In our terms it states that the conjugacy is  $C^{r-1-\delta-\varepsilon}$ -smooth for any  $\varepsilon > 0$  provided that  $0 < \delta < r - 2$ . Notice that Theorem 1 is stronger than the result just cited, though valid for a special scope of parameter values only, and it is sharp, i.e. smoothness of conjugacy higher than  $C^{r-1-\delta}$  cannot be achieved in general settings, as it

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follows from the examples constructed in [4]. At present, we do not know whether Theorem 1 can be extended further, and the examples mentioned do not prevent such an extension.

In paper by K. Khanin, Ya. Sinai [5], published simultaneously with [4], similar problems were approached by a different method. The method we use is different from the one of [4]; it is based on the ideas of [5], the cross-ratio distortion tools and certain exact relations between elements of the dynamically generated structure on the circle.

All the implicit constants in asymptotics written as  $\mathcal{O}(\cdot)$  depend on the function  $f$  only in Section 2 and on the diffeomorphism  $T$  only in Section 3.

## 2 Cross-ratio tools

The *cross-ratio* of four pairwise distinct points  $x_1, x_2, x_3, x_4$  is

$$\text{Cr}(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_3)(x_4 - x_1)}$$

Their *cross-ratio distortion* with respect to a strictly increasing function  $f$  is

$$\text{Dist}(x_1, x_2, x_3, x_4; f) = \frac{\text{Cr}(f(x_1), f(x_2), f(x_3), f(x_4))}{\text{Cr}(x_1, x_2, x_3, x_4)}$$

Clearly,

$$\text{Dist}(x_1, x_2, x_3, x_4; f) = \frac{D(x_1, x_2, x_3; f)}{D(x_1, x_4, x_3; f)}, \quad (2)$$

where

$$D(x_1, x_2, x_3; f) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} : \frac{f(x_2) - f(x_3)}{x_2 - x_3}$$

is the *ratio distortion* of three distinct points  $x_1, x_2, x_3$  with respect to  $f$ .

In the case of smooth  $f$  such that  $f'$  does not vanish, both the ratio distortion and the cross-ratio distortion are defined for points, which are not necessarily pairwise distinct, as the appropriate limits (or, just by formally replacing ratios  $(f(a) - f(a))/(a - a)$  with  $f'(a)$  in the definitions above).

Notice that both ratio and cross-ratio distortions are multiplicative with respect to composition: for two functions  $f$  and  $g$  we have

$$D(x_1, x_2, x_3; f \circ g) = D(x_1, x_2, x_3; g) \cdot D(g(x_1), g(x_2), g(x_3); f) \quad (3)$$

$$\text{Dist}(x_1, x_2, x_3, x_4; f \circ g) = \text{Dist}(x_1, x_2, x_3, x_4; g) \cdot \text{Dist}(g(x_1), g(x_2), g(x_3), g(x_4); f) \quad (4)$$

For  $f \in C^{3+\beta}$  it is possible to evaluate the next entry in the asymptotical expansions for both ratio and cross-ratio distortions. The *Swartz derivative* of  $C^{3+\beta}$ -smooth function is defined as  $Sf = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})$ .

**Proposition 1.** *Let  $f \in C^{3+\beta}$ ,  $\beta \in [0, 1]$ , and  $f' > 0$  on  $[A, B]$ . Then for any  $x_1, x_2, x_3 \in [A, B]$  the following estimate holds:*

$$D(x_1, x_2, x_3; f) = 1 + (x_1 - x_3) \left( \frac{f''(x_1)}{2f'(x_1)} + \frac{1}{6} Sf(x_1)(x_2 + x_3 - 2x_1) + \mathcal{O}(\Delta^{1+\beta}) \right), \quad (5)$$

where  $\Delta = \max\{x_1, x_2, x_3\} - \min\{x_1, x_2, x_3\}$ .

We start by proving the following

**Lemma 1.** *For arbitrary  $\theta \in [A, B]$  we have*

$$\begin{aligned} \frac{f''(\theta)}{2f'(\theta)} + \frac{f'''(\theta)}{6f'(\theta)}(x_1 + x_2 + x_3 - 3\theta) - \left( \frac{f''(\theta)}{2f'(\theta)} \right)^2 (x_2 + x_3 - 2\theta) = \\ \frac{f''(x_1)}{2f'(x_1)} + \frac{1}{6}Sf(x_1)(x_2 + x_3 - 2x_1) + \mathcal{O}(\Delta_\theta^{1+\beta}), \end{aligned} \quad (6)$$

where  $\Delta_\theta = \max\{x_1, x_2, x_3, \theta\} - \min\{x_1, x_2, x_3, \theta\}$ .

*Proof.* Obvious estimates  $f''(x_1) = f''(\theta) + f'''(\theta)(x_1 - \theta) + \mathcal{O}(|x_1 - \theta|^{1+\beta})$  and  $f'(x_1) = f'(\theta) + f''(\theta)(x_1 - \theta) + \mathcal{O}((x_1 - \theta)^2)$  imply that

$$\frac{f''(x_1)}{2f'(x_1)} = \frac{f''(\theta)}{2f'(\theta)} + \left( \frac{f'''(\theta)}{2f'(\theta)} - \frac{(f''(\theta))^2}{2(f'(\theta))^2} \right) (x_1 - \theta) + \mathcal{O}(\Delta_\theta^{1+\beta}) \quad (7)$$

On the other hand,  $Sf(x_1) = Sf(\theta) + \mathcal{O}(|x_1 - \theta|^\beta)$  and  $|x_2 + x_3 - 2x_1| \leq 2\Delta_\theta$ , hence

$$\frac{1}{6}Sf(x_1)(x_2 + x_3 - 2x_1) = \left( \frac{f'''(\theta)}{6f'(\theta)} - \frac{(f''(\theta))^2}{4(f'(\theta))^2} \right) (x_2 + x_3 - 2x_1) + \mathcal{O}(\Delta_\theta^{1+\beta}) \quad (8)$$

Adding (7) and (8) gives (6).  $\square$

*Remark 1.* Notice, that Lemma 1, in particular, provides an alternative, more general (though less memorable) formulation of Proposition 1 as we may choose  $\theta = x_2$ , or  $x_3$ , or any other point between  $\min\{x_1, x_2, x_3\}$  and  $\max\{x_1, x_2, x_3\}$  to get the same order  $\mathcal{O}(\Delta^{1+\beta})$  as in (5).

*Proof of Proposition 1.* Using  $x_2$  as the reference point for taking derivatives, we get

$$\begin{aligned} \frac{f(x_1) - f(x_2)}{x_1 - x_2} &= f'(x_2) + \frac{1}{2}f''(x_2)(x_1 - x_2) + \frac{1}{6}f'''(x_2)(x_1 - x_2)^2 + \mathcal{O}(|x_1 - x_2|^{2+\beta}), \\ \frac{f(x_2) - f(x_3)}{x_2 - x_3} &= f'(x_2) + \frac{1}{2}f''(x_2)(x_3 - x_2) + \frac{1}{6}f'''(x_2)(x_3 - x_2)^2 + \mathcal{O}(|x_3 - x_2|^{2+\beta}), \end{aligned}$$

and after dividing (in view of the expansion  $(1+t)^{-1} = 1 - t + t^2 + \mathcal{O}(t^3)$ ) obtain

$$\begin{aligned} D(x_1, x_2, x_3; f) &= 1 + (x_1 - x_3) \left[ \frac{f''(x_2)}{2f'(x_2)} + \frac{f'''(x_2)}{6f'(x_2)}(x_1 + x_3 - 2x_2) \right. \\ &\quad \left. - \left( \frac{f''(x_2)}{2f'(x_2)} \right)^2 (x_3 - x_2) \right] + \mathcal{O}(\Delta^{2+\beta}) \end{aligned} \quad (9)$$

In the case when  $x_2$  lies between  $x_1$  and  $x_3$ , the estimate (9) implies

$$\begin{aligned} D(x_1, x_2, x_3; f) &= 1 + (x_1 - x_3) \left[ \frac{f''(x_2)}{2f'(x_2)} + \frac{f'''(x_2)}{6f'(x_2)}(x_1 + x_3 - 2x_2) \right. \\ &\quad \left. - \left( \frac{f''(x_2)}{2f'(x_2)} \right)^2 (x_3 - x_2) + \mathcal{O}(\Delta^{1+\beta}) \right] \end{aligned} \quad (10)$$

It is not hard to notice that the expression in the square brackets here is exactly the subject of Lemma 1 with  $\theta = x_2$ , thus (5) is proven.

Suppose that  $x_1$  lies between  $x_2$  and  $x_3$ . Then the version of (5) for  $D(x_2, x_1, x_3; f)$  is proven. Also, the version of (9) for  $D(x_1, x_3, x_2; f)$  is proven. One can check the following exact relation takes place:

$$D(x_1, x_2, x_3; f) = 1 + \frac{x_1 - x_3}{x_2 - x_3} (D(x_2, x_1, x_3; f) - 1) D(x_1, x_3, x_2; f) \quad (11)$$

Substituting

$$D(x_2, x_1, x_3; f) - 1 = (x_2 - x_3) \left( \frac{f''(x_2)}{2f'(x_2)} + \frac{1}{6} S f(x_2) (x_1 + x_3 - 2x_2) + \mathcal{O}(\Delta^{1+\beta}) \right)$$

and

$$D(x_1, x_3, x_2; f) = 1 + (x_1 - x_2) \frac{f''(x_2)}{2f'(x_2)} + \mathcal{O}(\Delta^{1+\beta})$$

into (11), we get (10), and Lemma 1 again implies (5).

The case when  $x_3$  lies between  $x_1$  and  $x_2$  is similar to the previous one. The case when two or three among the points  $x_1, x_2$  and  $x_3$  coincide, all considerations above are valid with obvious alterations.  $\square$

**Proposition 2.** *Let  $f \in C^{3+\beta}$ ,  $\beta \in [0, 1]$ , and  $f' > 0$  on  $[A, B]$ . For any  $x_1, x_2, x_3, x_4 \in [A, B]$  the following estimate holds:*

$$\text{Dist}(x_1, x_2, x_3, x_4; f) = 1 + (x_1 - x_3) \left( \frac{1}{6} (x_2 - x_3) S f(\theta) + \mathcal{O}(\Delta^{1+\beta}) \right) \quad (12)$$

where  $\Delta = \max\{x_1, x_2, x_3, x_4\} - \min\{x_1, x_2, x_3, x_4\}$  and  $\theta$  is an arbitrary point between  $\min\{x_1, x_2, x_3, x_4\}$  and  $\max\{x_1, x_2, x_3, x_4\}$ .

*Proof.* Proposition 1 and Lemma 1 imply

$$D(x_1, x_2, x_3; f) = 1 + (x_1 - x_3) \left[ \frac{f''(\theta)}{2f'(\theta)} + \frac{f'''(\theta)}{6f'(\theta)} (x_1 + x_2 + x_3 - 3\theta) - \left( \frac{f''(\theta)}{2f'(\theta)} \right)^2 (x_2 + x_3 - 2\theta) + \mathcal{O}(\Delta^{1+\beta}) \right],$$

$$D(x_1, x_4, x_3; f) = 1 + (x_1 - x_3) \left[ \frac{f''(\theta)}{2f'(\theta)} + \frac{f'''(\theta)}{6f'(\theta)} (x_1 + x_4 + x_3 - 3\theta) - \left( \frac{f''(\theta)}{2f'(\theta)} \right)^2 (x_4 + x_3 - 2\theta) + \mathcal{O}(\Delta^{1+\beta}) \right]$$

Dividing the first expression by the second one accordingly to (2) in view of the formula  $(1+t)^{-1} = 1 - t + t^2 + \mathcal{O}(t^3)$ , we get (12).  $\square$

*Remark 2.* Obviously enough, the estimate (12) can be re-written as

$$\log \text{Dist}(x_1, x_2, x_3, x_4; f) = (x_1 - x_3) \left( \frac{1}{6} (x_2 - x_3) S f(\theta) + \mathcal{O}(\Delta^{1+\beta}) \right) \quad (13)$$

## 3 Circle diffeomorphisms

### 3.1 Preparations

For an orientation-preserving homeomorphism  $T$  of the unit circle  $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ , its *rotation number*  $\rho = \rho(T)$  is the value of the limit  $\lim_{i \rightarrow \infty} L_T^i(x)/i$  for a lift  $L_T$  of  $T$  from  $\mathbb{T}^1$  onto  $\mathbb{R}$ . It is known since Poincaré that rotation number is always defined (up to an additive integer) and does not depend on the starting point  $x \in \mathbb{R}$ . Rotation number  $\rho$  is irrational if and only if  $T$  has no periodic points. We restrict our attention in this paper to this case. The order of points on the circle for any trajectory  $\xi_i = T^i \xi_0$ ,  $i \in \mathbb{Z}$ , coincides with the order of points for the rigid rotation  $R_\rho$ . This fact is sometimes referred to as the *combinatorial equivalence* between  $T$  and  $R_\rho$ .

We use the *continued fraction* expansion for the (irrational) rotation number:

$$\rho = [k_1, k_2, \dots, k_n, \dots] = \frac{1}{k_1 + \frac{1}{k_2 + \frac{1}{\dots \frac{1}{k_n + \frac{1}{\dots}}}}} \in (0, 1) \quad (14)$$

which, as usual, is understood as a limit of the sequence of *rational convergents*  $p_n/q_n = [k_1, k_2, \dots, k_n]$ . The positive integers  $k_n$ ,  $n \geq 1$ , called *partial quotients*, are defined uniquely for irrational  $\rho$ . The mutually prime positive integers  $p_n$  and  $q_n$  satisfy the recurrent relation  $p_n = k_n p_{n-1} + p_{n-2}$ ,  $q_n = k_n q_{n-1} + q_{n-2}$  for  $n \geq 1$ , where it is convenient to define  $p_0 = 0$ ,  $q_0 = 1$  and  $p_{-1} = 1$ ,  $q_{-1} = 0$ .

Given a circle homeomorphism  $T$  with irrational  $\rho$ , one may consider a *marked trajectory* (i.e. the trajectory of a marked point)  $\xi_i = T^i \xi_0 \in \mathbb{T}^1$ ,  $i \geq 0$ , and pick out of it the sequence of the *dynamical convergents*  $\xi_{q_n}$ ,  $n \geq 0$ , indexed by the denominators of the consecutive rational convergents to  $\rho$ . We will also conventionally use  $\xi_{q_{-1}} = \xi_0 - 1$ . The well-understood arithmetical properties of rational convergents and the combinatorial equivalence between  $T$  and  $R_\rho$  imply that the dynamical convergents approach the marked point, alternating their order in the following way:

$$\xi_{q_{-1}} < \xi_{q_1} < \xi_{q_3} < \dots < \xi_{q_{2m+1}} < \dots < \xi_0 < \dots < \xi_{q_{2m}} < \dots < \xi_{q_2} < \xi_{q_0} \quad (15)$$

We define the  $n$ th *fundamental segment*  $\Delta^{(n)}(\xi)$  as the circle arc  $[\xi, T^{q_n} \xi]$  if  $n$  is even and  $[T^{q_n} \xi, \xi]$  if  $n$  is odd. If there is a marked trajectory, then we use the notations  $\Delta_0^{(n)} = \Delta^{(n)}(\xi_0)$ ,  $\Delta_i^{(n)} = \Delta^{(n)}(\xi_i) = T^i \Delta_0^{(n)}$ .

The iterates  $T^{q_n}$  and  $T^{q_{n-1}}$  restricted to  $\Delta_0^{(n-1)}$  and  $\Delta_0^{(n)}$  respectively are nothing else but two continuous components of the first-return map for  $T$  on the segment  $\Delta_0^{(n-1)} \cup \Delta_0^{(n)}$  (with its endpoints being identified). The consecutive images of  $\Delta_0^{(n-1)}$  and  $\Delta_0^{(n)}$  until their return to  $\Delta_0^{(n-1)} \cup \Delta_0^{(n)}$  cover the whole circle without overlapping (beyond their endpoints), thus forming the  $n$ th *dynamical partition*

$$\mathcal{P}_n = \{\Delta_i^{(n-1)}, 0 \leq i < q_n\} \cup \{\Delta_i^{(n)}, 0 \leq i < q_{n-1}\}$$

of  $\mathbb{T}^1$ . The endpoints of the segments from  $\mathcal{P}_n$  form the set

$$\Xi_n = \{\xi_i, 0 \leq i < q_{n-1} + q_n\}$$

Denote by  $\Delta_n$  the length of  $\Delta^{(n)}(\xi)$  for the rigid rotation  $R_\rho$ . Obviously enough,  $\Delta_n = |q_n \rho - p_n|$ . It is well known that  $\Delta_n \sim \frac{1}{q_{n+1}}$  (here ‘ $\sim$ ’ means ‘comparable’, i.e. ‘ $A \sim B$ ’ means ‘ $A = \mathcal{O}(B)$  and  $B = \mathcal{O}(A)$ ’), thus the Diophantine properties of  $\rho \in D_\delta$  can be equivalently expressed in the form:

$$\Delta_{n-1}^{1+\delta} = \mathcal{O}(\Delta_n) \quad (16)$$

We will also have in mind the universal exponential decay property

$$\frac{\Delta_n}{\Delta_{n-k}} \leq \frac{\sqrt{2}}{(\sqrt{2})^k}, \quad (17)$$

which follows from the obvious estimates  $\Delta_n \leq \frac{1}{2}\Delta_{n-2}$  and  $\Delta_n < \Delta_{n-1}$ .

In [1] it was shown that for any diffeomorphism  $T \in C^{2+\alpha}(\mathbb{T}^1)$ ,  $T' > 0$ ,  $\alpha \in [0, 1]$ , with irrational rotation number the following Denjoy-type inequality takes place:

$$(T^{q_n})'(\xi) = 1 + \mathcal{O}(\varepsilon_{n,\alpha}), \quad \text{where} \quad \varepsilon_{n,\alpha} = l_{n-1}^\alpha + \frac{l_n}{l_{n-1}} l_{n-2}^\alpha + \frac{l_n}{l_{n-2}} l_{n-3}^\alpha + \cdots + \frac{l_n}{l_0} \quad (18)$$

and  $l_m = \max_{\xi \in \mathbb{T}^1} |\Delta_m(\xi)|$ . Notice, that this estimate does not require any Diophantine conditions on  $\rho(T)$ .

Unfortunately, it is not possible to write down a corresponding stronger estimate for  $T \in C^{3+\beta}(\mathbb{T}^1)$ ,  $\beta \in [0, 1]$ , without additional assumptions. We will assume that the conjugacy is at least  $C^1$ -smooth:  $\phi \in C^{1+\gamma}(\mathbb{T}^1)$ ,  $\phi' > 0$ , with some  $\gamma \in [0, 1]$ . (Notice, that in conditions of Theorem 1 this assumption holds true with  $\gamma = 1 - \delta$  accordingly to [1], and our aim is to raise the value of  $\gamma$  to  $1 - \delta + \beta$ .)

This assumption is equivalent to the following one: an invariant measure generated by  $T$  has the positive density  $h = \phi' \in C^\gamma(\mathbb{T}^1)$ . This density satisfies the homologic equation

$$h(\xi) = T'(\xi)h(T\xi) \quad (19)$$

The continuity of  $h$  immediately implies that  $h(\xi) \sim 1$ , and therefore  $(T^i)'(\xi) = \frac{h(\xi)}{h(T^i\xi)} \sim 1$  and

$$|\Delta^{(n)}(\xi)| \sim l_n \sim \Delta_n \sim \frac{1}{q_{n+1}}$$

(due to  $\Delta_n = \int_{\Delta^{(n)}(\xi)} h(\eta) d\eta$ ). By this reason, we introduce the notation

$$E_{n,\sigma} = \sum_{k=0}^n \frac{\Delta_n}{\Delta_{n-k}} \Delta_{n-k-1}^\sigma,$$

so that  $\varepsilon_{n,\alpha}$  in (18) can be replaced by  $E_{n,\alpha}$  as soon as we know of the existence of continuous  $h$ .

It follows also that  $(T^i)' \in C^\gamma(\mathbb{T}^1)$  uniformly in  $i \in \mathbb{Z}$ , i.e.

$$(T^i)'(\xi) - (T^i)'(\eta) = \mathcal{O}(|\xi - \eta|^\gamma), \quad (20)$$

since  $(T^i)' \xi - (T^i)' \eta = \frac{h(\xi)}{h(T^i \xi)} - \frac{h(\eta)}{h(T^i \eta)}$  and  $T^i \xi - T^i \eta \sim \xi - \eta$ .

The additional smoothness of  $T$  will be used through the following quantities:  $p_n = p_n(\xi_0) = \sum_{i=0}^{q_n-1} \frac{ST(\xi_i)}{h(\xi_i)} (\xi_i - \xi_{i+q_{n-1}})$ ,  $\bar{p}_n = \bar{p}_n(\xi_0) = \sum_{i=0}^{q_n-1} \frac{ST(\xi_{i+q_n})}{h(\xi_{i+q_n})} (\xi_{i+q_n} - \xi_i)$ . We have

$$p_n + \bar{p}_n = \sum_{\xi \in \Xi_n} ST(\hat{\xi}) \frac{\hat{\xi} - \xi}{h(\hat{\xi})}, \quad (21)$$

where  $\hat{\xi}$  denotes the point from the set  $\Xi_n$  following  $\xi$  in the (circular) order  $\dots \rightarrow \xi_{q_{n-1}} \rightarrow \xi_0 \rightarrow \xi_{q_n} \rightarrow \dots$ . It is easy to see that  $N_n(\xi_i) = \xi_{i+q_n}$  for  $0 \leq i < q_{n-1}$  and  $N_n(\xi_i) = \xi_{i-q_{n-1}}$  for  $q_{n-1} \leq i < q_n + q_{n-1}$ .

In the next two subsections, we will establish certain dependencies between the Denjoy-type estimates in the forms  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(\Delta_n^\nu)$  and  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(E_{n,\sigma})$ .

### 3.2 Statements that use the Hoelder exponents of $T'''$ and $h$

In all the statements of this subsection, we assume that  $T \in C^{3+\beta}$  and  $h \in C^\gamma$ ,  $\beta, \gamma \in [0, 1]$ , but do not make any use of Diophantine properties of  $\rho$ .

The next lemma corresponds to the exact integral relation  $\int_{\mathbb{T}^1} \frac{ST(\xi)}{h(\xi)} d\xi$  first demonstrated in [5].

**Lemma 2.** *If  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(\Delta_n^\nu)$ , then  $p_n + \bar{p}_n = \mathcal{O}(\Delta_{n-1}^{\min\{\beta, 2\nu-1\}})$ .*

*Proof.* Using the representation  $ST = \left(\frac{T''}{T'}\right)' - \frac{1}{2} \left(\frac{T''}{T'}\right)^2$ , from (21) we derive

$$\begin{aligned} p_n + \bar{p}_n &= \sum_{\xi \in \Xi_n} \left[ \left( \frac{T''(\hat{\xi})}{T'(\hat{\xi})} - \frac{T''(\xi)}{T'(\xi)} \right) \frac{1}{h(\hat{\xi})} + \mathcal{O}(|\hat{\xi} - \xi|^{1+\beta}) \right] \\ &\quad - \frac{1}{2} \sum_{\xi \in \Xi_n} \left( \frac{T''(\xi)}{T'(\xi)} \right)^2 \frac{\hat{\xi} - \xi}{h(\hat{\xi})} \\ &= \sum_{\xi \in \Xi_n} \frac{T''(\xi)}{T'(\xi)} \left[ \frac{1}{h(\xi)} - \frac{1}{h(\hat{\xi})} - \frac{1}{2} \frac{T''(\xi)}{T'(\xi)} \frac{\hat{\xi} - \xi}{h(\hat{\xi})} \right] + \mathcal{O}(\Delta_{n-1}^\beta) \end{aligned}$$

Notice that

$$h(\xi) - h(\hat{\xi}) = \mathcal{O}(|\hat{\xi} - \xi|^\nu) \quad (22)$$

due to (19). In particular, (22) implies that the expression in the last square brackets is  $\mathcal{O}(|\hat{\xi} - \xi|^\gamma)$ , hence using the estimate  $T''(\xi) = \frac{T'(\hat{\xi}) - T'(\xi)}{\hat{\xi} - \xi} + \mathcal{O}(\hat{\xi} - \xi)$  we get

$$p_n + \bar{p}_n = \sum_{\xi \in \Xi_n} \left( \frac{T'(\hat{\xi})}{T'(\xi)} - 1 \right) \frac{1}{\hat{\xi} - \xi} \left[ \frac{1}{h(\xi)} - \frac{1}{h(\hat{\xi})} - \frac{1}{2} \left( \frac{T'(\hat{\xi})}{T'(\xi)} - 1 \right) \frac{1}{h(\hat{\xi})} \right] + \mathcal{O}(\Delta_{n-1}^{\min\{\beta, \nu\}})$$

Now, the substitutions  $T'(\xi) = \frac{h(\xi)}{h(T\xi)}$  and  $T'(\hat{\xi}) = \frac{h(\hat{\xi})}{h(T\hat{\xi})}$  transform the last estimate (exactly) into

$$p_n + \bar{p}_n = \frac{1}{2} \sum_{\xi \in \Xi_n} \frac{h(\hat{\xi})}{(h(\xi))^2 (\hat{\xi} - \xi)} \left[ \left( \frac{h(\xi)}{h(\hat{\xi})} - 1 \right)^2 - \left( \frac{h(T\xi)}{h(T\hat{\xi})} - 1 \right)^2 \right] + \mathcal{O}(\Delta_{n-1}^{\min\{\beta, \nu\}}) \quad (23)$$

Similarly to (22), each one of two expressions in parentheses here are  $\mathcal{O}(|\hat{\xi} - \xi|^\nu)$ . It follows, firstly, that

$$p_n + \bar{p}_n = \frac{1}{2} \sum_{\xi \in \Xi_n} \left( \frac{h(T\xi)}{h(T\hat{\xi})} - 1 \right)^2 \left[ \frac{h(T\hat{\xi})}{(h(T\xi))^2(T\hat{\xi} - T\xi)} - \frac{h(\hat{\xi})}{(h(\xi))^2(\hat{\xi} - \xi)} \right] + \mathcal{O}(\Delta_{n-1}^{\min\{\beta, 2\nu-1\}}), \quad (24)$$

since, as it is easy to see, the sums in (23) and in (24) differ by a finite number of terms of the order  $\mathcal{O}(|\hat{\xi} - \xi|^{2\nu-1})$ , and  $2\nu - 1 \leq \nu$ . Secondly, we have

$$\frac{h(T\hat{\xi})}{(h(T\xi))^2(T\hat{\xi} - T\xi)} : \frac{h(\hat{\xi})}{(h(\xi))^2(\hat{\xi} - \xi)} - 1 = \frac{T'(\xi)}{T'(\hat{\xi})} \cdot \left( T'(\xi) : \frac{T\hat{\xi} - T\xi}{\hat{\xi} - \xi} \right) - 1 = \mathcal{O}(\hat{\xi} - \xi),$$

so the expressions in the square brackets in (24) are bounded, and therefore the whole sum in it is  $\sum_{\xi \in \Xi_n} \mathcal{O}(|\hat{\xi} - \xi|^{2\nu}) = \mathcal{O}(\Delta_{n-1}^{2\nu-1})$ .  $\square$

Notice, that Lemma 2 does not use  $\gamma$ . However, the next one does.

**Lemma 3.** *If  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(\Delta_n^\nu)$ , then  $p_n = \mathcal{O}(\Delta_{n-1}^{\min\{\beta, 2\nu-1, \gamma\}})$ .*

*Proof.* It follows from (20) that

$$\frac{|\Delta_i^{(n)}|}{|\Delta_0^{(n)}|} : \frac{|\Delta_i^{(n-2)}|}{|\Delta_0^{(n-2)}|} = 1 + \mathcal{O}(\Delta_{n-2}^\gamma) \quad (25)$$

This implies, together with (22) and  $ST(\xi_{i+q_n}) - ST(\xi_i) = \mathcal{O}(\Delta_n^\beta)$ , that

$$\bar{p}_n + \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-2)}|} p_{n-1} = \sum_{i=0}^{q_{n-1}-1} \mathcal{O}(\Delta_n(\Delta_{n-2}^\gamma + \Delta_n^\beta + \Delta_n^\nu)) = \frac{\Delta_n}{\Delta_{n-2}} \mathcal{O}(\Delta_{n-2}^{\min\{\beta, \gamma, \nu\}}) = \mathcal{O}(\Delta_n^{\min\{\beta, \gamma, \nu\}})$$

In view of this, Lemma 2 implies  $p_n = \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-2)}|} p_{n-1} + \mathcal{O}(\Delta_{n-1}^\mu)$ , where  $\mu = \min\{\beta, 2\nu - 1, \gamma\} \leq 1$ . Telescoping the last estimate, we get

$$p_n = \sum_{k=0}^n \frac{|\Delta_0^{(n)}| \cdot |\Delta_0^{(n-1)}|}{|\Delta_0^{(n-k)}| \cdot |\Delta_0^{(n-k-1)}|} \mathcal{O}(\Delta_{n-k-1}^\mu) = \mathcal{O} \left( \Delta_{n-1}^\mu \sum_{k=0}^n \frac{\Delta_n}{\Delta_{n-k}} \left( \frac{\Delta_{n-1}}{\Delta_{n-k-1}} \right)^{1-\mu} \right),$$

and the latter sum is bounded due to (17).  $\square$

**Lemma 4.** *If  $p_n = \mathcal{O}(\Delta_{n-1}^\omega)$ , where  $\omega \in [0, 1]$ , then*

$$\begin{aligned} \text{Dist}(\xi_0, \xi, \xi_{q_{n-1}}, \eta; T^{q_n}) &= 1 + (\xi - \eta) \mathcal{O}(\Delta_{n-1}^{\min\{\beta, \gamma, \omega\}}), \quad \xi, \eta \in \Delta_0^{(n-1)}; \\ \text{Dist}(\xi_0, \xi, \xi_{q_n}, \eta; T^{q_{n-1}}) &= 1 + (\xi - \eta) \frac{\Delta_n}{\Delta_{n-2}} \mathcal{O}(\Delta_{n-2}^{\min\{\beta, \gamma, \omega\}}), \quad \xi, \eta \in \Delta_0^{(n-2)} \end{aligned}$$



*Proof.* Accordingly to (13) and (4), we have

$$\log \text{Dist}(\xi_0, \xi, \xi_{q_{n-1}}, \eta; T^{q_n}) = \frac{1}{6} \sum_{i=0}^{q_n-1} (\xi_i - \xi_{i+q_{n-1}})(T^i \xi - T^i \eta) ST(\xi_i) + (\xi - \eta) \mathcal{O}(\Delta_{n-1}^\beta)$$

On the other hand,

$$\begin{aligned} & \sum_{i=0}^{q_n-1} (\xi_i - \xi_{i+q_{n-1}})(T^i \xi - T^i \eta) ST(\xi_i) - h(\xi_0)(\xi - \eta) p_n \\ &= (\xi - \eta) \sum_{i=0}^{q_n-1} (\xi_i - \xi_{i+q_{n-1}}) ST(\xi_i) \left[ \frac{T^i \xi - T^i \eta}{\xi - \eta} - (T^i)'(\xi_0) \right] = (\xi - \eta) \mathcal{O}(\Delta_{n-1}^\gamma) \end{aligned}$$

because of (20). The first estimate of the lemma follows. To prove the second one, we similarly notice that

$$\log \text{Dist}(\xi_0, \xi, \xi_{q_n}, \eta; T^{q_{n-1}}) = \frac{1}{6} \sum_{i=0}^{q_{n-1}-1} (\xi_i - \xi_{i+q_n})(T^i \xi - T^i \eta) ST(\xi_i) + (\xi - \eta) \mathcal{O}(\Delta_{n-1}^\beta)$$

and

$$\begin{aligned} & \sum_{i=0}^{q_{n-1}-1} (\xi_i - \xi_{i+q_n})(T^i \xi - T^i \eta) ST(\xi_i) - h(\xi_0)(\xi - \eta) \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-2)}|} p_{n-1} \\ &= (\xi - \eta) \sum_{i=0}^{q_{n-1}-1} (\xi_i - \xi_{i+q_n}) ST(\xi_i) \left[ \frac{T^i \xi - T^i \eta}{\xi - \eta} - (T^i)'(\xi_0) \frac{|\Delta_i^{(n-2)}|}{|\Delta_0^{(n-2)}|} : \frac{|\Delta_i^{(n)}|}{|\Delta_0^{(n)}|} \right] \\ &= (\xi - \eta) \sum_{i=0}^{q_{n-1}-1} (\xi_i - \xi_{i+q_n}) ST(\xi_i) \mathcal{O}(\Delta_{n-2}^\gamma) = (\xi - \eta) \frac{\Delta_n}{\Delta_{n-2}} \mathcal{O}(\Delta_{n-2}^\gamma) \end{aligned}$$

(see (25)). □

As in [1], we introduce the functions

$$M_n(\xi) = D(\xi_0, \xi, \xi_{q_{n-1}}; T^{q_n}), \quad \xi \in \Delta_0^{(n-1)};$$

$$K_n(\xi) = D(\xi_0, \xi, \xi_{q_n}; T^{q_{n-1}}), \quad \xi \in \Delta_0^{(n-2)},$$

where  $\xi_0$  is arbitrarily fixed. The following three exact relations can be easily checked:

$$M_n(\xi_0) \cdot M_n(\xi_{q_{n-1}}) = K_n(\xi_0) \cdot K_n(\xi_{q_n}), \tag{26}$$

$$K_{n+1}(\xi_{q_{n-1}}) - 1 = \frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n-1)}|} (M_n(\xi_{q_{n+1}}) - 1), \tag{27}$$

$$\frac{(T^{q_{n+1}})'(\xi_0)}{M_{n+1}(\xi_0)} - 1 = \frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n)}|} \left( 1 - \frac{(T^{q_n})'(\xi_0)}{K_{n+1}(\xi_0)} \right) \tag{28}$$

Also notice that

$$\frac{M_n(\xi)}{M_n(\eta)} = \text{Dist}(\xi_0, \xi, \xi_{q_{n-1}}, \eta; T^{q_n}), \quad \frac{K_n(\xi)}{K_n(\eta)} = \text{Dist}(\xi_0, \xi, \xi_{q_n}, \eta; T^{q_{n-1}}) \quad (29)$$

**Lemma 5.** *If  $p_n = \mathcal{O}(\Delta_{n-1}^\omega)$ ,  $\omega \in [0, 1]$ , then  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(E_{n, 1+\min\{\beta, \gamma, \omega\}})$ .*

*Proof.* Let  $\sigma = 1 + \min\{\beta, \gamma, \omega\}$ . In view of (29), Lemma 4 implies that  $M_n(\xi)/M_n(\eta) = 1 + \mathcal{O}(\Delta_{n-1}^{\sigma+1})$  and  $K_n(\xi)/K_n(\eta) = 1 + \mathcal{O}(\Delta_n \Delta_{n-2}^\sigma)$ . In our assumptions, the functions  $M_n(\xi) \sim 1$  and  $K_n(\xi) \sim 1$ , since  $(T^i)'(\xi) \sim 1$ . This gives us

$$M_n(\xi) = m_n + \mathcal{O}(\Delta_{n-1}^{\sigma+1}), \quad K_n(\xi) = m_n + \mathcal{O}(\Delta_n \Delta_{n-2}^\sigma) \quad (30)$$

where  $m_n^2$  denotes the products in (26). Due to (27) and (30) we have

$$m_{n+1} - 1 = \frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n-1)}|} (m_n - 1) + \mathcal{O}(\Delta_{n+1} \Delta_{n-1}^\sigma), \quad (31)$$

which is telescoped into  $m_n - 1 = \mathcal{O}(\Delta_n E_{n-1, \sigma-1})$ , which in turn implies

$$M_n(\xi) = 1 + \mathcal{O}(\Delta_{n-1} E_{n, \sigma-1}), \quad K_n(\xi) = 1 + \mathcal{O}(\Delta_n E_{n-1, \sigma-1}) \quad (32)$$

(notice that  $\Delta_{n-1} E_{n, \sigma-1} = \Delta_{n-1}^{1+\sigma} + \Delta_n E_{n-1, \sigma-1}$ ). Due to (27) and (32) we have

$$(T^{q_{n+1}})'(\xi_0) - 1 = \frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n)}|} (1 - (T^{q_n})'(\xi_0)) + \mathcal{O}(\Delta_n E_{n+1, \sigma-1}) \quad (33)$$

which is telescoped into

$$\begin{aligned} (T^{q_n})'(\xi_0) - 1 &= \mathcal{O} \left( \sum_{k=0}^n \frac{\Delta_n}{\Delta_{n-k}} \Delta_{n-k-1} E_{n-k, \sigma-1} \right) \\ &= \mathcal{O} \left( \Delta_n \sum_{k=0}^n \sum_{m=0}^{n-k} \frac{\Delta_{n-k-1}}{\Delta_{n-k-m}} \Delta_{n-k-m-1}^\sigma \right) = \mathcal{O} \left( \Delta_n \sum_{k=0}^n \sum_{s=k}^n \frac{\Delta_{n-k-1}}{\Delta_{n-s}} \Delta_{n-s-1}^\sigma \right) \\ &= \mathcal{O} \left( \Delta_n \sum_{s=0}^n \frac{\Delta_{n-s-1}^\sigma}{\Delta_{n-s}} \sum_{k=0}^s \Delta_{n-k-1} \right) = \mathcal{O}(E_{n, \sigma}), \end{aligned}$$

since  $\sum_{k=0}^s \Delta_{n-k-1} = \mathcal{O}(\Delta_{n-s-1})$  due to (17).  $\square$

The summary of this subsection is given by

**Proposition 3.** *Suppose that for a diffeomorphism  $T \in C^{3+\beta}(\mathbb{T}^1)$ ,  $T' > 0$ ,  $\beta \in [0, 1]$ , with irrational rotation number there exists density  $h \in C^\gamma(\mathbb{T}^1)$ ,  $\gamma \in [0, 1]$ , of the invariant measure and the following asymptotical estimate holds true:  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(\Delta_n^\nu)$  with certain real constant  $\nu$ . Then  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(E_{n, 1+\min\{\beta, \gamma, 2\nu-1\}})$ .*

*Proof.* Follows from Lemmas 3 and 5 immediately.  $\square$

*Remark 3.* In [3] it is shown that for any  $T \in C^3(\mathbb{T}^1)$  the following Denjoy-type estimate takes place:  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(l_n^{1/2})$ , and in our assumptions it is equivalent to  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(\Delta_n^{1/2})$ . Hence, in fact we have  $\nu \geq \frac{1}{2}$ , though this is of no use for us.

### 3.3 Statements that use Diophantine properties of $\rho$

Now we start using the assumption  $\rho \in D_\delta$ ,  $\delta \geq 0$ , however forget about the smoothness of  $T$  and the Hoelder condition on  $h$ .

**Lemma 6.** *If  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(\Delta_n^\nu)$ ,  $\nu \in [\frac{\delta}{1+\delta}, 1]$ , then  $h \in C^{\nu(1+\delta)-\delta}(\mathbb{T}^1)$ .*

*Proof.* Consider two points  $\xi_0, \xi \in \mathbb{T}^1$  and  $n \geq 0$  such that  $\Delta_n \leq |\phi(\xi) - \phi(\xi_0)| < \Delta_{n-1}$ . Let  $k$  be the greatest positive integer such that  $|\phi(\xi) - \phi(\xi_0)| \geq k\Delta_n$ . (It follows that  $1 \leq k \leq k_{n+1}$ .) Due to the combinatorics of trajectories, continuity of  $h$  and the homologic equation (19), we have

$$\log h(\xi) - \log h(\xi_0) = \mathcal{O} \left( k\Delta_n^\nu + \sum_{s=n+1}^{+\infty} k_{s+1}\Delta_s^\nu \right),$$

and the same estimate holds for  $h(\xi) - h(\xi_0)$ , since  $\log h(\xi) = \mathcal{O}(1)$ .

We have  $k_{n+1} < \Delta_{n-1}/\Delta_n = \mathcal{O}(\Delta_n^{-\frac{\delta}{1+\delta}})$ , hence

$$k\Delta_n^\nu = k^{\nu(1+\delta)-\delta} \Delta_n^{\nu(1+\delta)-\delta} \cdot k^{(1+\delta)(1-\nu)} \Delta_n^{\delta(1-\nu)} = \mathcal{O}((k\Delta_n)^{\nu(1+\delta)-\delta})$$

and

$$\sum_{m=n+1}^{+\infty} k_{m+1}\varepsilon_m = \mathcal{O} \left( \sum_{m=n+1}^{+\infty} \Delta_m^{\frac{\nu(1+\delta)-\delta}{1+\delta}} \right) = \mathcal{O} \left( \sum_{m=n+1}^{+\infty} \Delta_{m-1}^{\nu(1+\delta)-\delta} \right) = \mathcal{O}(\Delta_n^{\nu(1+\delta)-\delta})$$

due to (16) and (17). Finally, we obtain

$$h(\xi) - h(\xi_0) = \mathcal{O}((k\Delta_n)^{\nu(1+\delta)-\delta}) = \mathcal{O}(|\phi(\xi) - \phi(\xi_0)|^{\nu(1+\delta)-\delta}) = \mathcal{O}(|\xi - \xi_0|^{\nu(1+\delta)-\delta})$$

□

**Lemma 7.** *If  $\sigma \in [0, 1 + \delta)$ , then  $E_{n,\sigma} = \mathcal{O}(\Delta_n^{\frac{\sigma}{1+\delta}})$ .*

*Proof.* Due to (16) we have

$$E_{n,\sigma} = \mathcal{O} \left( \Delta_n \sum_{k=0}^n \Delta_{n-k}^{\frac{\sigma}{1+\delta}-1} \right)$$

The statement of the lemma follows, since  $\sum_{k=0}^n \Delta_{n-k}^{\frac{\sigma}{1+\delta}-1} = \mathcal{O}(\Delta_n^{\frac{\sigma}{1+\delta}-1})$  because of (17). □

This subsection is summarized by

**Proposition 4.** *Suppose that for a diffeomorphism  $T \in C^1(\mathbb{T}^1)$ ,  $T' > 0$ , with rotation number  $\rho \in D_\delta$ ,  $\delta \geq 0$ , there exists a continuous density  $h$  of the invariant measure, and the following asymptotical estimate holds true:  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(E_{n,\sigma})$  with certain constant  $\sigma \in [0, 1 + \delta)$ . Then  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(\Delta_n^{\frac{\sigma}{1+\delta}})$  and  $h \in C^{\max\{0, \sigma-\delta\}}(\mathbb{T}^1)$ .*

*Proof.* Follows from Lemmas 7 and 6 immediately. □

### 3.4 Proof of Theorem 1

Recall that we need to prove Theorem 1 for  $r = 3 + \beta$ ,  $0 < \beta < \delta < 1$ . We will use a finite inductive procedure based on Propositions 3 and 4 to improve step by step the Denjoy-type estimate in the form

$$(T^{q_n})'(\xi) = 1 + \mathcal{O}(E_{n,\sigma}) \quad (34)$$

From [1], it follows that (34) holds true for  $\sigma = 1$  (see (18)), so this will be our starting point. Consider the sequence  $\sigma_0 = 1$ ,  $\sigma_{i+1} = \min \left\{ 1 + \beta, \frac{2}{1+\delta} \sigma_i \right\}$ ,  $i \geq 0$ . The inductive step is given by the following

**Lemma 8.** *Suppose that  $\sigma_i \in [1, 1 + \beta]$  and (34) holds for  $\sigma = \sigma_i$ . Then  $\sigma_{i+1} \in [1, 1 + \beta]$  and (34) holds for  $\sigma = \sigma_{i+1}$ .*

*Proof.* First of all, notice that  $\sigma_i < 1 + \delta$  since  $\beta < \delta$ . Proposition 4 implies that  $h \in C^{\gamma_i}(\mathbb{T}^1)$  with  $\gamma_i = \sigma_i - \delta \in (0, 1)$  and  $(T^{q_n})'(\xi) = 1 + \mathcal{O}(\Delta_n^{\nu_i})$  with  $\nu_i = \frac{\sigma_i}{1+\delta} \in (0, 1)$ . Proposition 3 then implies that (34) holds for  $\sigma = \min\{1 + \beta, 1 + \gamma_i, 2\nu_i\}$ , and this is exactly  $\sigma_{i+1}$  since  $1 + \sigma_i - \delta > \frac{2\sigma_i}{1+\delta}$  (indeed,  $(1 + \sigma_i - \delta)(1 + \delta) - 2\sigma_i = (1 - \delta)(1 + \delta - \sigma_i) > 0$ ). The bounds on  $\sigma_{i+1}$  are easy to derive.  $\square$

What is left is to notice that  $\sigma_i = \min \left\{ 1 + \beta, \left( \frac{2}{1+\delta} \right)^i \right\}$ ,  $i \geq 0$ , where  $\frac{2}{1+\delta} > 1$ , so this sequence reaches  $1 + \beta$  in a finite number of steps. And as soon as (34) with  $\sigma = 1 + \beta$  is shown, Proposition 4 implies that  $h \in C^{1+\beta-\delta}$ . Theorem 1 is proven.

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### References

1. K. Khanin and A. Teplinsky. Herman's theory revisited, [arXiv:math.DS/0707.0075](https://arxiv.org/abs/math/0707.0075).
2. M.-R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle a des rotations, *I. H. E. S. Publ. Math.*, **49**, 5–233 (1979).
3. J.-C. Yoccoz. Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation verifie une condition diophantienne. *Ann. Sci. Ecole Norm. Sup. (4)*, **17**:3, 333–359 (1984).
4. Y. Katznelson and D. Ornstein. The differentiability of the conjugation of certain diffeomorphisms of the circle, *Ergodic Theory Dynam. Systems*, **9**:4, 643–680 (1989).
5. Ya. G. Sinai and K. M. Khanin, Smoothness of conjugacies of diffeomorphisms of the circle with rotations, *Uspekhi Mat. Nauk* **44**:1, 57–82 (1989), in Russian; English transl., *Russian Math. Surveys* **44**:1, 69–99 (1989).